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Fourier Expansions with Respect to the Rayleigh System

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We give conditions for a function f in order that its Fourier series with respect to the Rayleigh system $\{e^{a(x)\sqrt{n^2-\omega^2}} \sin nx\}_{n=1}^{\infty}$ be convergent uniformly on the interval $[\varepsilon, \pi]$, $\varepsilon > 0$. © 2000 Academic Press

1. INTRODUCTION AND NOTATION

There are many papers (see, e.g., [4, 5, 7]) devoted to the study of the properties of the system of the form $\{e^{na(x)} \sin nx\}_{n=1}^{\infty}$ as well as of the more general systems $\{e^{a(x)\sqrt{n^2-\omega^2}} \sin nx\}_{n=1}^{\infty}$ (Rayleigh system). An important property of these systems is their completeness and minimality in $L^p(0, \pi)$ $(1 < p < \infty)$. (A short history can be found in [5].) There are at least two reasons that the Rayleigh systems are of interest:

1. They occur as model systems in many questions about the root vector of differential operator pencils.

2. They occur in considering the classical Rayleigh problem about scattering the plane monochromatic wave on a periodic surface given by the equation $y = a(x)$ (see [6]).

In treating plane wave scattering by a periodic surface, Lord Rayleigh assumed that the discrete outgoing and evanescent plane wave representation for the scattered field was valid on the surface itself. Recently this Rayleigh assumption has been questioned and criticized. It turned out that it was true in some cases, but false in the general case.

For solving the corresponding boundary problem for the Helmholz equation $\Delta u + \omega^2 u = 0$ it is necessary to describe the class of functions that admit the series expansion with respect to the system $\{e^{a(x)\sqrt{n^2-\omega^2}} \sin nx\}_{n=1}^{\infty}$.

In the case of the system $\{e^{-np(x)} \sin nx, e^{-np(x)} \cos nx\}_{n=0}^{\infty}$ the completeness in $L^2(0, 2\pi)$ was proved in [7] under the assumption $p \in C^{1,\alpha}[0, 2\pi]$ (the derivative of p satisfies a Hölder condition of order α).

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In this paper we will consider the case when $a=a(x)$ is an analytic function in a neighborhood of $[0, \pi]$ (i.e., in an open set containing $[0, \pi]$) and describe the class of functions with the Fourier series (with respect to the Rayleigh system) uniformly convergent on $[\varepsilon, \pi]$, $(0<\varepsilon<\pi)$.

Let

$$y_n(x)=e^{a(x)\sqrt{n^2-\omega^2}}\sin nx$$

$$\varphi_n(x)=e^{na(x)}\sin nx, \qquad n=1, 2, \ldots .$$

In what follows we will assume that $a=a(x)$ is an analytic function in a neighborhood of $[0, \pi]$ such that $a(0)=0$, $a'(0)<0$ and $a(x)<0$ for $x\in(0, \pi]$.

Let

$$L_0=\{(x, a(x)) : 0\leqslant x\leqslant\pi\}$$

$$L_1=\{(0, y) : y\geqslant 0\}$$

$$L_2=\{(\pi, y) : y\geqslant a(\pi)\}$$

$$\Omega=\{(x, y) : 0\leqslant x\leqslant\pi,\ y\geqslant a(x)\}.$$

By $\langle f, g\rangle$ we denote the inner product in L^2, i.e.,

$$\langle f, g\rangle=\int_0^\pi f(x)\,\overline{g(x)}\,dx.$$

2. THE CONSTRUCTION OF THE SYSTEM BIORTHOGONAL TO $\{\varphi_n\}$ AND ITS NORM ESTIMATE (THE CASE $\omega=0$)

Let $\mathscr{D}(A)=\{f : f\in C[0, \pi], f(0)=f(\pi)=0\}$. For $f\in\mathscr{D}(A)$ define the function

$$U_f(x, y)=\begin{cases}0; & (x, y)\in L_1\\ 0; & (x, y)\in L_2.\\ f(x); & (x, y)\in L_0\end{cases}$$

Obviously U_f is a continuous function on $\partial\Omega(=L_0\cup L_1\cup L_2)$. Consider the Dirichlet problem for Ω:

$$\begin{cases}\Delta u=0 & \text{on } \Omega\\ u|_{\partial\Omega}=U_f.\end{cases} \tag{1}$$

Since U_f is continuous on $\partial\Omega$, then solution u_f of the problem (1) is a continuous function on $\bar{\Omega}$ (see [3]). Define the operator

$$A\colon \mathscr{D}(A) \to L^2(0, \pi)$$

by

$$Af(x) = u_f(x, 0).$$

It follows that $Af \in C[0, \pi]$. Denote by $\pi\theta_1$ and $\pi\theta_2$ the inner angles with respect to Ω between L_1 and L_0 and between L_2 and L_0, respectively, i.e., $\pi\theta_1 = \frac{\pi}{2} - \arctan a'(0)$, $\pi\theta_2 = \frac{\pi}{2} + \arctan a'(\pi)$. It is clear that $0 < \theta_2 < 1$. Since $a'(0) < 0$ it follows that $\frac{1}{2} < \theta_1 < 1$. Let $w = h(z)$ be the conformal mapping of the upper half plane onto Ω and $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$ and $h(\alpha) = 0$, $h(\beta) = \pi + ia(\pi)$.

According to the results of Warschawski (see [9, 10]) there hold the formulas

$$h(z) \sim \text{const}(z-\alpha)^{\theta_1}, \qquad |h'(z)| \asymp |z-\alpha|^{\theta_1 - 1}, \qquad z \to \alpha \tag{2}$$

$$h(z) - (\pi + ia(\pi)) \sim \text{const}(z-\beta)^{\theta_2}, \qquad |h'(z)| \asymp |z-\beta|^{\theta_2 - 1}, \qquad z \to \beta. \tag{3}$$

Let h^{-1} be the inverse function of h. Then

$$g(z) = \frac{h^{-1}(z) - i}{h^{-1}(z) + i}$$

maps conformaly the domain Ω onto the unit disc. Then (see 8) the solution to the problem (1) is given by

$$u_f(x, y) = \operatorname{Re}\left(\frac{1}{2\pi i}\int_{\partial\Omega} U_f(\xi)\,\frac{g(\xi)+g(z)}{g(\xi)-g(z)}\cdot\frac{g'(\xi)}{g(\xi)}\,d\xi\right), \qquad z = x + iy.$$

From this and the definition of U_f we get

$$u_f(x, y) = \operatorname{Re}\left(\frac{1}{2\pi i}\int_0^{\pi} f(t)\,\frac{g(t+ia(t))+g(z)}{g(t+ia(t))-g(z)}\cdot\frac{g'(t+ia(t))}{g(t+ia(t))}\cdot(1+ia'(t))\,dt\right)$$

and hence

$$Af(x) = u_f(x, 0) = \operatorname{Re}\left(\frac{1}{2\pi i}\int_0^{\pi} f(t)\,\frac{g(t+ia(t))+g(x)}{g(t+ia(t))-g(x)}\cdot\frac{g'(t+ia(t))}{g(t+ia(t))}\cdot(1+ia'(t))\,dt\right).$$

Let

$$K(x, t) = \operatorname{Re}\left(\frac{1}{2\pi i}\frac{g(t+ia(t))+g(x)}{g(t+ia(t))-g(x)}\cdot\frac{g'(t+ia(t))}{g(t+ia(t))}\cdot(1+ia'(t))\right).$$

If $f \in \mathscr{D}(A)$ is a real-valued function, then

$$Af(x) = \int_0^\pi K(x, t)\, f(t)\, dt.$$

The functions $\varphi_m(t) = e^{ma(t)} \sin mt$ belong to $\mathscr{D}(A)$ and are real-valued, and therefore

$$A\varphi_m(x) = \int_0^\pi K(x, t)\, \varphi_m(t)\, dt.$$

On the other hand it is clear that

$$A\varphi_m = e^{my} \sin mx|_{y=0} = \sin mx.$$

Since $\{\sqrt{2/\pi}\sin nx\}_{n=1}^{\infty}$ is an orthonormal base of $L^2(0, \pi)$ we get

$$\frac{2}{\pi}\langle \sin nx, \sin mx\rangle = \delta_{nm},$$

i.e.,

$$\left\langle \frac{2}{\pi}\sin nx, A\varphi_m\right\rangle = \delta_{nm},$$

and hence

$$\left\langle \frac{2}{\pi}A^* \sin nx, \varphi_m\right\rangle = \delta_{nm} \qquad (A^* \text{ is the adjoint operator of } A).$$

Let

$$\Psi_n(x) = \frac{2}{\pi}A^* \sin nx \left(= \frac{2}{\pi}\int_0^\pi K(t, x) \sin nt\, dt\right).$$

Thus the system $\{\varphi_n\}_1^\infty$ is minimal and its biorthogonal system is $\{\Psi_n\}_1^\infty$. Let us show that $\Psi_n \in L^2(0, \pi)$ and estimate its norm. First prove the following lemma:

LEMMA 1. *We have*

(a) $\int_0^\pi \int_0^\pi t^2 \, |K(t, x)|^2 \, dx \, dt < \infty$

(b) $\int_0^\pi \int_0^\pi t^2 \, |K(x, t)|^2 \, dx \, dt < \infty.$

Proof. (a) Since $|g(x+ia(x))| = 1$ and $|g(t)| \leqslant 1$ and a' is a continuous function, we get

$$|K(t, x)| \leqslant C_1 \frac{|g'(x+ia(x))|}{|g(x+ia(x)) - g(t)|}.$$

Having in mind that

$$g'(z) = 2i \frac{(h^{-1}(z))'}{(i+h^{-1}(z))^2} \qquad \text{and} \qquad h([\alpha, \beta]) = L_0$$

we have

$$|g'(x+ia(x))| \leqslant 2 \, |(h^{-1}(z))'| = \frac{2}{|h'(h^{-1}(z))|}, \qquad z = x + ia(x).$$

Since $h^{-1}(0) = \alpha$ the behavior of the function $h'(h^{-1}(z))$ near $z = 0$ is the same as the behavior of $h'(z)$ near $z = \alpha$.

Because of (2) the function $1/|h'(h^{-1}(z))|$ is bounded near $z = 0$. In a similar way we get that it is bounded near $z = \pi + ia(\pi)$. From this it follows that

$$\sup_{x \in [0, \pi]} |g'(x+ia(x))| < +\infty.$$

Thus

$$|K(t, x)| \leqslant C_2 \frac{1}{|g(x+ia(x)) - g(t)|}$$

(C_2 is independent of x, t). From the relations

$$|g(x+ia(x)) - g(t)| = 2 \frac{|h^{-1}(x+ia(x)) - h^{-1}(t)|}{(h^{-1}(t)+i) \cdot (h^{-1}(x+ia(x))+i)}$$

and

$$\sup_{x, t \in [0, \pi]} |i + h^{-1}(x+ia(x))| \cdot |i + h^{-1}(t)| < +\infty$$

we obtain

$$|K(t, x)| \leqslant C_3 \frac{1}{|h^{-1}(x+ia(x))-h^{-1}(t)|} \tag{4}$$

(C_3 is independent of x, t). From (4), by integration, we get

$$\int_0^{\pi} |K(t, x)|^2 \, dx \leqslant C_3^2 \int_0^{\pi} \frac{dx}{|h^{-1}(x+ia(x))-h^{-1}(t)|^2}.$$

After the change $u = h^{-1}(x+ia(x))$ we get

$$\int_0^{\pi} |K(t, x)|^2 \, dx \leqslant C_3^2 \int_{\alpha}^{\beta} \frac{|h'(u)| \, du}{|u-h^{-1}(t)|^2}$$

and this shows that it is enough to estimate the last integral when $t \to 0^+$. From (2) it follows

$$\begin{aligned} \int_{\alpha}^{\beta} \frac{|h'(u)| \, du}{|u-h^{-1}(t)|^2} &\leqslant \text{const} \int_{\alpha}^{\beta} \frac{|u-\alpha|^{\theta_1 - 1}}{|u-h^{-1}(t)|^2} \, du \\ &= \text{cont} \int_0^{\beta-\alpha} \frac{s^{\theta_1 - 1} \, ds}{|s-(h^{-1}(t)-\alpha)|^2}. \end{aligned} \tag{5}$$

Again from (2),

$$b(t) \stackrel{\text{def}}{=} h^{-1}(t) - \alpha \sim \left(\frac{t}{c_0}\right)^{1/\theta_1}, \qquad t \to 0^+ \quad (c_0 = \text{const}),$$

and, because of

$$\int_0^{\beta-\alpha} \frac{s^{\theta_1 - 1} \, ds}{|s-b(t)|^2} = O\left(\frac{(b(t))^{\theta_1 - 1} - (\overline{b(t)})^{\theta_1 - 1}}{b(t) - \overline{b(t)}}\right), \qquad t \to 0^+,$$

we have

$$\int_0^{\beta-\alpha} \frac{s^{\theta_1 - 1} \, ds}{|s-b(t)|^2} = O\left(\frac{t^{(1/\theta_1)(\theta_1 - 1)}}{t^{1/\theta_1}}\right) = O(t^{1-2/\theta_1}), \qquad t \to 0^+.$$

The last asymptotic formula and (5) give

$$\int_0^{\pi} |K(t, x)|^2 \, dx = O(t^{1-2/\theta_1}), \qquad t \to 0^+.$$

From this, we get

$$\int_0^\pi t^2\,dt \int_0^\pi |K(t,x)|^2\,dx \leqslant \text{const} \int_0^\pi t^{3-2/\theta_1} < +\infty$$

because $\frac{1}{2} < \theta_1$. This concludes the proof of (a).
Assertion (b) is proved in a similar way. ∎

LEMMA 2. *For the system* $\{\Psi_n\}_{n=1}^\infty$ *there holds*

$$\|\Psi_n\| \leqslant C\sqrt{n},$$

where C *is independent of* n.

Proof. From

$$\Psi_n(x) = \frac{2}{\pi} \int_0^\pi K(t,x) \sin nt\,dt$$

it follows that

$$\begin{aligned} |\Psi_n(x)|^2 &\leqslant \frac{4}{\pi^2} \int_0^\pi t^2\,|K(t,x)|^2\,dt \int_0^\pi \frac{\sin^2 nt}{t^2}\,dt \\ &= \frac{4}{\pi^2}\, n \int_0^{n\pi} \frac{\sin^2 t}{t^2}\,dt \cdot \int_0^\pi t^2\,|K(t,x)|^2\,dt \\ &\leqslant \frac{4}{\pi^2}\, n \int_0^\infty \frac{\sin^2 t}{t^2}\,dt \cdot \int_0^\pi t^2\,|K(t,x)|^2\,dt. \end{aligned}$$

Hence we get

$$\|\Psi_n\|^2 \leqslant \frac{4}{\pi^2}\, n \int_0^\infty \frac{\sin^2 t}{t^2}\,dt \cdot \int_0^\pi \int_0^\pi t^2\,|K(t,x)|^2\,dx\,dt.$$

Lemma 2 follows from the last inequality and Lemma 1. ∎

Now, let $f \in L^2(0,\pi)$ be a function such that its Fourier series with respect to $\{\varphi_n\}_{n=1}^\infty$ is pointwise convergent,

$$f(x) = \sum_{n=1}^\infty \langle f, \Psi_n \rangle\, \varphi_n(x).$$

From Lemma 2 we see that for every fixed ε $(0<\varepsilon<\pi)$, there holds the inequality

$$|\langle f, \Psi_n\rangle \cdot \varphi_n(x)| \leqslant C\sqrt{n}\, \|f\| \exp(n \cdot \max_{x\in[\varepsilon,\pi]} a(x))$$

and because of the properties of $a=a(x)$, the series $\sum_{n=1}^{\infty} \langle f, \Psi_n\rangle\, \varphi_n(x)$ is uniformly convergent on $[\varepsilon, \pi]$ for every $\varepsilon>0$. It follows that f is continuous on $(0, \pi]$. it is clear that

$$f(0)=f(\pi)=0.$$

Since a is analytic in a neightborhood of $[0, \pi]$ it follows that the series

$$\sum_{n=1}^{\infty} \langle f, \Psi_n\rangle\, \varphi_n(x).$$

converges uniformly in a neighborhood of the interval $[\varepsilon, \pi]$ for every $\varepsilon>0$. Therefore the sum of the series $\sum_{n=1}^{\infty} \langle f, \Psi_n\rangle\, \varphi_n(x)$ is a function analytic in neighborhood of $(0, \pi]$. Also, we have $f(0)=f(\pi)=0$. Let us show that these conditions together with the condition $\lim_{x\to 0^+} f(x)=0$ imply the converse.

THEOREM 1. *Let $f\in\mathscr{D}(A)$ and let f have an analytic continuation to a domain containing $(0, \pi]$. Then*

$$f(x)=\sum_{n=1}^{\infty} \langle f, \Psi_n\rangle\, \varphi_n(x)$$

and for every $\varepsilon>0$ the series converges uniformly on $[\varepsilon, \pi]$.

For the proof we need a lemma.

LEMMA 3. *If $f\in\mathscr{D}(A)$, then $\sum_{n=1}^{\infty} \langle f, \Psi_n\rangle\, \varphi_n(x)\in\mathscr{D}(A)$.*

Proof. It suffices to prove that

$$\lim_{x\to 0^+} \sum_{n=1}^{\infty} \langle f, \Psi_n\rangle\, \varphi_n(x)=0.$$

Since $f\in\mathscr{D}(A)$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \langle f, \Psi_n\rangle\, \varphi_n(x) &= \frac{2}{\pi}\sum_{n=1}^{\infty} \langle Af, \sin ny\rangle \cdot \varphi_n(x)\\ &= \frac{1}{\pi}\int_0^{\pi} Af(y) \sum_{n=0}^{\infty} e^{na(x)}(\cos n(x-y)-\cos n(x+y))\, dy.\end{aligned} \tag{6}$$

Let

$$H(x, y) = \frac{1}{1-\exp(a(x)+i(x-y))} + \frac{1}{a(x)+i(x-y)} + \frac{1}{1-\exp(a(x)-i(x-y))} + \frac{1}{a(x)-i(x-y)} - \frac{1}{1-\exp(a(x)+i(x+y))} - \frac{1}{a(x)+i(x+y)} - \frac{1}{1-\exp(a(x)-i(x+y))} - \frac{1}{a(x)-i(x+y)}.$$

From (6) it follows that

$$\sum_{n=1}^{\infty} \langle f, \Psi_n \rangle \varphi_n(x) = \frac{1}{2\pi} \int_0^\pi Af(y)\, H(x, y)\, dy + \frac{1}{\pi} \int_0^\pi Af(y) \cdot a(x) \cdot \left[\frac{1}{a^2(x)+(x+y)^2} - \frac{1}{a^2(x)+(x-y)^2} \right] dy.$$

Since $f \in \mathscr{D}(A)$, we have $Af \in C[0, \pi]$ and $(Af)(0) = 0(= u_f(0, 0))$. Having in mind that $\frac{H(x, y)}{x}$ is a bounded function on $[0, \pi]^2$, we obtain

$$\lim_{x \to 0^+} \frac{1}{2\pi} \int_0^\pi Af(y)\, H(x, y)\, dy = 0$$

and this reduces the proof of Lemma 3 to

$$\lim_{x \to 0^+} \int_0^\pi Af(y) \left(\frac{1}{a^2(x)+(x+y)^2} - \frac{1}{a^2(x)+(x-y)^2} \right) a(x)\, dy = 0. \tag{7}$$

Since $a(0) = 0$ and $a = a(x)$ is analytic near $x = 0$, a direct verification shows that

$$\lim_{x \to 0^+} \int_0^\pi a(x)\, Af(y) \left(\frac{1}{a^2(x)+(x \pm y)^2} - \frac{1}{a'^2(0)\, x^2+(x \pm y)^2} \right) dy = 0,$$

which reduces the proof of (7) to

$$\lim_{x \to 0^+} \int_0^\pi a(x)\, Af(y) \left(\frac{1}{a'^2(0)\, x^2+(x+y)^2} - \frac{1}{a'^2(0)\, x^2+(x-y)^2} \right) dy = 0,$$

i.e., (because $a(0)=0,\ a'(0)\neq 0$),

$$\lim_{x\to 0^+}\int_0^\pi |Af(y)|\,\frac{yx^2}{(c_0x^2+(x+y)^2)(c_0x^2+(x-y)^2)}\,dy=0, \qquad c_0=a'^2(0). \tag{8}$$

Let $\varepsilon>0$. Since $Af\in C[0,\pi]$ and $Af(0)=0$, there exists $\delta>0$ such that $|Af(y)|<\varepsilon$ for $0<y<\delta$. Since

$$\begin{aligned}\int_0^\pi |Af(y)|\,&\frac{yx^2}{(c_0x^2+(x+y)^2)(c_0x^2+(x-y)^2)}\,dy\\ &=\int_0^\delta |Af(y)|\,\frac{yx^2}{(c_0x^2+(x+y)^2)(c_0x^2+(x-y)^2)}\,dy\\ &\quad+\int_\delta^\pi |Af(y)|\,\frac{yx^2}{(c_0x^2+(x+y)^2)(c_0x^2+(x-y)^2)}\,dy,\end{aligned}$$

$$\begin{aligned}\int_0^\delta |Af(y)|\,&\frac{yx^2}{(c_0x^2+(x+y)^2)(c_0x^2+(x-y)^2)}\,dy,\\ &<\varepsilon\cdot\int_0^\delta \frac{yx^2}{(c_0x^2+(x+y)^2)(c_0x^2+(x-y)^2)}\,dy\\ &=\varepsilon\cdot\int_0^{\delta/x}\frac{t\,dt}{(c_0+(1+t)^2)(c_0+(1-t)^2)}<D_0\cdot\varepsilon,\end{aligned}$$

where

$$D_0=\int_0^\infty \frac{t\,dt}{(c_0+(1+t)^2)(c_0+(1-t)^2)}$$

and

$$\lim_{x\to 0^+}\int_\delta^\pi |Af(y)|\,\frac{yx^2}{(c_0x^2+(x+y)^2)(c_0x^2+(x-y)^2)}\,dy=0$$

(by the Lebesgue dominated convergence theorem), we obtain (8), which was to be proved. ∎

Proof of Theorem 1. Let f satisfy the conditions of Theorem 1. Let

$$r(x)=f(x)-\sum_{n=1}^{\infty}\langle f,\Psi_n\rangle\,\varphi_n(x).$$

By Lemma 3, the function r belongs to the $\mathscr{D}(A)$. Since

$$\langle r, \Psi_n \rangle = \langle f, \Psi_n \rangle - \langle f, \Psi_n \rangle = 0, \qquad n = 1, 2, 3, \ldots$$

and $r \in \mathscr{D}(A)$, we obtain

$$\langle Ar, \sin nx \rangle = 0, \qquad n = 1, 2, 3, \ldots .$$

Because of completeness of the system $\{\sin nx\}_1^\infty$ in $L^2(0, \pi)$, we have $Ar = 0$ a.e. on $[0, \pi]$. But, since $r \in \mathscr{D}(A)$, it follows that $Ar \in C[0, \pi]$ and hence

$$Ar \equiv 0 \qquad \text{on } [0, \pi].$$

Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0 \quad \text{on } \Omega \\ u|_{\partial\Omega} = U_r = \begin{cases} 0; & (x, y) \in L_1 \\ 0; & (x, y) \in L_2. \\ r(x); & (x, y) \in L_0 \end{cases} \end{cases} \tag{9}$$

Since $r \in \mathscr{D}(A)$ we have $U_r \in C(\partial\Omega)$ and hence

$$Ar = u_r(x, 0),$$

where u_r is the solution to (9).

The condition $Ar \equiv 0$ reduces to the $u_r(x, 0) = 0$ for $0 \leqslant x \leqslant \pi$. Since the function u_r is harmonic in Ω and $u_r(x, 0) = 0$ $(0 \leqslant x \leqslant \pi)$ and $u_r(0, y) = u_r(\pi, y) = 0$ $(y \geqslant 0)$, it follows that

$$u_r(x, y) \equiv 0 \qquad \text{for} \quad (x, y) \in \Omega_1 = \{(x, y) \colon 0 \leqslant x \leqslant \pi,\ y \geqslant 0\}.$$

Hence $u_r(x, y) \equiv 0$ on Ω and hence $u_r(x, y) = 0$ on $\partial\Omega$. This implies

$$r(x) = u_r(x, a(x)) \equiv 0 \qquad \text{on } [0, \pi].$$

This proves Theorem 1. ∎

3. THE CASE WHEN ω IS SMALL

Let

$$y_n(x) = e^{a(x)\sqrt{n^2 - \omega^2}} \sin nx, \qquad n = 1, 2, 3, \ldots,$$

and let f satisfy the conditions of Theorem 1. Define the operator S by

$$Sf=\sum_{n=1}^{\infty}\langle f, \Psi_n\rangle\, y_n.$$

It is clear that polynomials vanishing at the ends of $[0, \pi]$ are dense in $L^2(0, \pi)$ and satisfy the hypothesis of Theorem 1. Since

$$f=\sum_{n=1}^{\infty}\langle f, \Psi_n\rangle\, \varphi_n,$$

provided that f satisfies the conditions of Theorem 1, we have

$$(S-I)\, f=\sum_{n=1}^{\infty}\langle f, \Psi_n\rangle\cdot(y_n-\varphi_n).$$

LEMMA 4. *The operator $I-S$ extends to a bounded operator on $L^2(0, \pi)$. Moreover the extension is a compact operator.*

Proof. Let f be as in Theorem 1. Since

$$\varphi_n(x)-y_n(x)=\varphi_n(x)\cdot\left(\frac{\omega^2 a(x)}{n+\sqrt{n^2-\omega^2}}+\frac{h_n(x)}{n^2}\right),$$

where $\sup_n \max_{x\in[0,\pi]}|h_n(x)|=M<\infty$, we get

$$(I-S)\, f=T_1 f+T_2 f,$$

where

$$T_1 f(x)=\sum_{n=1}^{\infty}\langle f, \Psi_n\rangle\, \varphi_n(x)\, \frac{\omega^2 a(x)}{n+\sqrt{n^2-\omega^2}}$$

and

$$T_2 f(x)=\sum_{n=1}^{\infty}\langle f, \Psi_n\rangle\, \varphi_n(x)\, \frac{h_n(x)}{n^2}.$$

Since by Lemma 2

$$\begin{aligned}\left\|T_2-\sum_{n=1}^{N}\langle\cdot, \Psi_n\rangle\, \varphi_n \frac{h_n}{n^2}\right\| &\leqslant \sum_{n=N+1}^{\infty}\|\Psi_n\|\,\|\varphi_n\|\,\frac{M}{n^2}\\ &\leqslant \text{const}\sum_{n=N+1}^{\infty} n^{-3/2}\to 0, \qquad N\to\infty,\end{aligned}$$

we see that T_2 is a compact operator.

It remains to prove that the operator T_1 is compact. This reduces to proving that the operator T_3 defined by

$$T_3 f = \sum_{n=1}^{\infty} \langle f, \Psi_n \rangle \frac{\varphi_n}{n}$$

is compact.

Let $f \in \mathscr{D}(A)$. Then

$$\begin{aligned} T_3 f(x) &= \frac{1}{\pi} \int_0^{\pi} Af(y) \cdot \sum_{n=1}^{\infty} \frac{2 \sin ny \sin nx}{n} e^{na(x)} \, dy \\ &= \frac{1}{\pi} \int_0^{\pi} H_1(x, y) \, Af(y) \, dy + \frac{1}{2\pi} \int_0^{\pi} Af(y) \ln \frac{a^2(x) + (x+y)^2}{a^2(x) + (x-y)^2} \, dy, \end{aligned}$$

where

$$\begin{aligned} H_1(x, y) = &-\frac{1}{2} \ln \frac{1 - \exp(a(x) + i(x-y))}{-a(x) - i(x-y)} - \frac{1}{2} \ln \frac{1 - \exp(a(x) - i(x-y))}{-a(x) + i(x-y)} \\ &+ \frac{1}{2} \ln \frac{1 - \exp(a(x) + i(x+y))}{-a(x) - i(x+y)} + \frac{1}{2} \ln \frac{1 - \exp(a(x) - i(x+y))}{-a(x) + i(x+y)}. \end{aligned}$$

Let

$$T'_3 f(x) = \frac{1}{\pi} \int_0^{\pi} H_1(x, y) \, Af(y) \, dy$$

$$T''_3 f(x) = \frac{1}{2\pi} \int_0^{\pi} Af(y) \ln \frac{a^2(x) + (x+y)^2}{a^2(x) + (x-y)^2} \, dy.$$

Then $T_3 = T'_3 + T''_3$. Let us show that T'_3 and T''_3 are compact operators. The kernel of T'_3 is

$$\mathscr{H}_1(x, t) = \frac{1}{\pi} \int_0^{\pi} H_1(x, y) \, K(y, t) \, dy.$$

Since

$$|\mathscr{H}_1(x, t)|^2 \leqslant \frac{1}{\pi^2} \int_0^{\pi} \frac{|H_1(x, y)|^2}{y^2} \, dy \int_0^{\pi} y^2 \, |K(y, t)|^2 \, dy,$$

we get by integration

$$\int_0^{\pi} \int_0^{\pi} |\mathscr{H}_1(x, t)|^2 \, dx \, dt \leqslant \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \frac{|H_1(x, y)|^2}{y^2} \, dx \, dy \cdot \int_0^{\pi} \int_0^{\pi} y^2 \, |K(y, t)|^2 \, dy \, dt.$$

Having in mind that $\frac{H_1(x, y)}{y}$ is bounded on $[0, \pi]^2$ we get from Lemma 1(a) and the preceding inequality

$$\int_0^\pi \int_0^\pi |\mathcal{H}_1(x, t)|^2 \, dx \, dt < \infty,$$

i.e., the operator T_3' is compact (and moreover Hilbert–Schmidt).

The kernel T_3'' is

$$\mathcal{H}_2(x, t) = \frac{1}{2\pi} \int_0^\pi \ln \frac{a^2(x) + (x + y)^2}{a^2(x) + (x - y)^2} K(y, t) \, dy.$$

This implies that

$$|\mathcal{H}_2(x, t)| \leqslant \frac{2}{\pi} \int_0^\pi |K(y, t)| \frac{xy}{a^2(x) + (x - y)^2} \, dy$$

(because $\ln(1 + s) \leqslant s$, $s \geqslant 0$). Since

$$\min_{x \in [0, \pi]} \frac{a^2(x)}{x^2} = d_0 > 0$$

it follows from the last inequality

$$|\mathcal{H}_2(x, t)| \leqslant \frac{2}{\pi} \int_0^\pi |K(y, t)| \frac{xy}{d_0 x^2 + (x - y)^2} \, dy \tag{10}$$

and since the function

$$R(x, y) = \frac{x}{d_0 x^2 + (x - y)^2}$$

is homogeneous of order -1 we have, by the Hardy–Littlewood inequality (see 2)

$$\int_0^\pi dx \left| \int_0^\pi R(x, y) \, \varphi(y) \, dy \right|^2 \leqslant \left(\int_0^\infty R(1, u) \, u^{-1/2} \, du \right)^2 \int_0^\pi |\varphi(y)|^2 \, dy,$$

where $\varphi(y) = y \, |K(y, t)|$, i.e.,

$$\int_0^\pi dx \left| \int_0^\pi K(y, t) \frac{xy \, dy}{d_0 x^2 + (x - y)^2} \right|^2 \leqslant \left(\int_0^\infty \frac{u^{-1/2} \, du}{d_0 + (1 - u)^2} \right)^2 \int_0^\pi y^2 \, |K(y, t|^2 \, dy.$$

Integrating the last inequality from $t=0$ to $t=\pi$, then using (10) and Lemma 1(a), we obtain

$$\int_0^\pi \int_0^\pi |\mathscr{H}_2(x, t)|^2 \, dx \, dt < \infty.$$

Hence, T_3'' extends to a Hilbert–Schmidt operator on $L^2(0, \pi)$. This completes the proof of Lemma 4. ∎

Using Lemma 4 and Lemma 1(b) one easily proves the following

LEMMA 5. *The operator $A(I-S)$ extends to a bounded operator on $L^2(0, \pi)$.*

Let

$$\sum_{n=1}^{\infty} \|\Psi_n\| \cdot \|y_n - \varphi_n\| = q(\omega).$$

Hence, using Taylor series of $1-e^z$, $z = a(x)(\sqrt{n^2-\omega^2} - n)$ we obtain

$$q(\omega) \leqslant \omega^2 C(\omega), \tag{11}$$

where $C(\omega)$ is a function bounded near $\omega = 0$. If f satisfies the hypotheses of Theorem 1, then

$$\|(I-S)\, f\| \leqslant q(\omega)\, \|f\|.$$

Denote the extension of $I-S$ and S to all $L^2(0, \pi)$ by the same letters. From (11) it follows that there exists an interval $[0, \omega_0]$ such that

$$q_0 = \sup_{\omega \in [0, \omega_0]} q(\omega) < 1.$$

Hence

$$\|I-S\| \leqslant q_0 < 1 \qquad \text{on } [0, \omega_0],$$

which implies that the operator S is invertible.

LEMMA 6. *Let f satisfy the conditions of Theorem 1 and let $\omega \in [0, \omega_0]$. Then the function $(S^{-1}f)(x)$ can be continued analytically to some domain containing the interval $(0, \pi]$ and $S^{-1}f$ belongs to $\mathscr{D}(A)$, i.e., $S^{-1}f$ satisfies the conditions of Theorem 1.*

Proof. Let f satisfy the hypotheses of Theorem 1. It is easily seen that

$$(I-S)\, f(x) = \sum_{n=1}^{\infty} \langle f, \Psi_n \rangle \cdot \varphi_n(x) \cdot \frac{a(x)}{n} \cdot g_n(x),$$

where g_n are functions analytic and uniformly bounded in some neighborhood of $[0, \pi]$. But, then $(I-S) f \in \mathscr{D}(A)$ and $(I-S) f$ has an analytic continuation to some neighborhood of $(0, \pi]$. It follows that the same property is possessed by all the functions $(I-S)^m f$ $(m=1, 2, 3, ...)$. For $\omega \in [0, \omega_0]$ the operator S is invertible and

$$S^{-1}f(x)=f(x)+(I-S)\, f(x)+(I-S)^2\, f(x)\cdots.$$

This series is norm convergent in $L^2(0, \pi)$. Let us show that it converges uniformly in some neighborhood of $(0, \pi]$. Since

$$\begin{aligned}(I-S)^m f &= (I-S)(I-S)^{m-1} f\\ &= a(x) \sum_{n=1}^{\infty} \langle (I-S)^{m-1} f, \Psi_n \rangle \frac{\varphi_n(x)\, g_n(x)}{n}\end{aligned}$$

and $(I-S)^{m-1} f \in \mathscr{D}(A)$ $(m=1, 2, 3, ...)$, we obtain

$$(I-S)^m f=\sqrt{\frac{2}{\pi}}\, a(x) \sum_{n=1}^{\infty} \left\langle A(I-S)(I-S)^{m-2} f, \sqrt{\frac{2}{\pi}} \sin nx \right\rangle \cdot \frac{\varphi_n(x) \cdot g_n(x)}{n}.$$

Applying the inequality of Cauchy and Bessel we get

$$|(I-S)^m f(x)|^2 \leqslant \frac{2}{\pi} |a(x)|^2 \sum_{n=1}^{\infty} \frac{|\varphi_n(x) \cdot g_n(x)|^2}{n^2} \cdot \|A(I-S)(I-S)^{m-2} f\|^2.$$

Since $\sup_{n, x} |g_n(x)| = M_1 < \infty$ and $\|I-S\| \leqslant q_0 < 1$ we get, by Lemma 5,

$$|(I-S)^m f(x)|^2 \leqslant \frac{2}{\pi} M_1^2\, |a(x)|^2 \sum_{n=1}^{\infty} \frac{|\varphi_n(x)|^2}{n^2} \cdot \|A(I-S)\|^2\, q_0^{2m-4}\, \|f\|^2,$$

i.e.,

$$\begin{aligned}|(I-S)^m f(x)| &\leqslant \sqrt{\frac{2}{\pi}} M_1\, |a(x)| \left(\sum_{n=1}^{\infty} \frac{\varphi_n(x)}{n^2} \right)^{1/2} \|A(I-S)\|\, q_0^{m-2}\, \|f\|\\ &\leqslant \sqrt{\frac{2}{\pi}} M_1\, |a(x)| \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \|A(I-S)\|\, q_0^{m-2}\, \|f\|.\end{aligned}$$

From this it follows that the series

$$\sum_{m=0}^{\infty} (I-S)^m f(x)$$

converges uniformly on $[0, \pi]$ and since $(I-S)^m f \in \mathscr{D}(A)$ it follows that $\sum_{m=0}^{\infty} (I-S)^m f \in \mathscr{D}(A)$.

An inspection of the above proof shows that the series

$$\sum_{m=0}^{\infty} (I-S)^m f(z)$$

converges uniformly in some neighborhood of $[\varepsilon, \pi]$ for every $\varepsilon > 0$ and therefore the sum is an analytic function in some neighborhood of $(0, \pi]$. Hence, the function f satisfies the conditions of Theorem 1. ∎

Let $\omega \in [0, \omega_0]$. Then S^{-1} is a bounded operator. Set $\theta_n = S^{*-1}\Psi_n$ ($\Rightarrow \Psi_n = S^*\theta_n$). From the definition of S it follows that $S\varphi_n = y_n$ and hence

$$\langle y_n, \theta_m \rangle = \langle S\varphi_n, \theta_m \rangle = \langle \varphi_n, S^*\theta_m \rangle = \langle \varphi_n, \Psi_m \rangle = \delta_{nm}.$$

Therefore the system $\{\theta_n\}$ is biorthogonal to $\{y_n\}_{n=1}^{\infty}$.

THEOREM 2. *If the function f satisfies the conditions of Theorem* 1 *and* $\omega \in [0, \omega_0]$, *then*

$$f(x) = \sum_{n=1}^{\infty} \langle f, \theta_n \rangle \, y_n(x), \tag{12}$$

where the series converges uniformly on every interval $[\varepsilon, \pi]$, $0 < \varepsilon < \pi$.

Proof. From

$$\|\theta_n\| = \|S^{*-1}\Psi_n\| \leqslant \|S^{*-1}\| \cdot \|\Psi_n\|$$

and Lemma 2 it follows that

$$\|\theta_n\| = O(\sqrt{n}).$$

An immediate consequence is that the series

$$\sum_{n=1}^{\infty} \langle f, \theta_n \rangle \, y_n(x)$$

converges uniformly on $[\varepsilon, \pi]$, $0 < \varepsilon < \pi$. Consider the function

$$r_1(x) = f(x) - \sum_{n=1}^{\infty} \langle f, \theta_n \rangle \, y_n(x) \left(= f(x) - \sum_{n=1}^{\infty} \langle S^{-1}f, \Psi_n \rangle \, y_n(x) \right).$$

Since the function $S^{-1}f$ has the same properties as f the sum of the series $\sum_{n=1}^{\infty} \langle S^{-1}f, \Psi_n \rangle y_n$ belongs to $\mathscr{D}(A)$ (because $\sum_1^{\infty} \langle S^{-1}f, \Psi_n \rangle \varphi_n \in C[0, \pi]$, which is a consequence of Lemma 3 and $\sum \langle S^{-1}f, \Psi_n \rangle \cdot (y_n - \varphi_n) \in C[0, \pi]$). Hence $r_1 \in C[0, \pi]$. It follows that

$$S^{-1}r_1 = S^{-1}f - \sum_1^{\infty} \langle S^{-1}f, \Psi_n \rangle S^{-1}y_n$$
$$= S^{-1}f - \sum_1^{\infty} \langle S^{-1}f, \Psi_n \rangle \varphi_n.$$

Since $S^{-1}f \in \mathscr{D}(A)$, we have by Theorem 1 that $S^{-1}r_1 = 0$ in L^2, i.e., $r_1 = 0$ a.e. on $[0, \pi]$. Since r_1 is continuous we see that

$$r_1 \equiv 0 \qquad \text{on } [0, \pi].$$

This concludes the proof of the theorem. ∎

4. THE GENERAL CASE

Lemma 4 shows that the operator

$$K(\omega^2) = I - S = \sum_{n=1}^{\infty} \langle \cdot, \Psi_n \rangle (\varphi_n - y_n)$$
$$= \sum_{n=1}^{\infty} \langle \cdot, \Psi_n \rangle (e^{na(x)} \sin nx - e^{a(x)\sqrt{n^2 - \omega^2}} \sin nx)$$

is compact.

Let $\omega^2 = \lambda$, $S = I - K(\lambda)$,

$$K(\lambda) = \sum_{n=1}^{\infty} \langle \cdot, \Psi_n \rangle (e^{na(x)} - e^{a(x)\sqrt{n^2 - \lambda}}) \sin nx,$$

$$\Omega_1 = \mathbb{C} \backslash \{\lambda\colon \operatorname{Re} \lambda = n^2, \operatorname{Im} \lambda < 0, n = 1, 2, 3...\},$$

$$\sqrt{n^2 - \lambda} = |n^2 - \lambda|^{1/2} e^{(i/2) \arg (n^2 - \lambda)}, \qquad -\frac{\pi}{2} < \arg (n^2 - \lambda) < \frac{3\pi}{2}.$$

The function $K(\lambda)$ is an analytic operator function on Ω_1 (with values in the set of compact operators). Hence, by the known theorem (see [1, Thm. I.5.1]) it follows that there exists a set $\mathscr{S}$ of isolated points in Ω_1 such that the operator $S = I - K(\lambda)$ is invertible for every $\lambda \in \Omega_1 \backslash \mathscr{S}$. It follows that if $\lambda \notin \mathscr{S}$ then the system $\{\theta_n\}_{n=1}^{\infty}$ $(\theta_n = (((I - K(\lambda))^{-1})^* \Psi_n))$ is biorthogonal to the system $\{e^{a(x)\sqrt{n^2 - \lambda}} \sin nx\}_{n=1}^{\infty}$.

THEOREM 3. *Let f satisfy the conditions of Theorem* 1 *and let* $\omega^2 \in \mathbb{R}$ *be such that* $\omega^2 \notin \mathcal{S}$. *Then*

$$f(x) = \sum_{n=1}^{\infty} \langle f, \theta_n \rangle \, y_n(x)$$

and the series converges uniformly on every $[\varepsilon, \pi]$ $(0 < \varepsilon < \pi)$.

Proof. For a fixed x, $(0 < x \leqslant \pi)$ let

$$G(\lambda) = f(x) - \sum_{n=1}^{\infty} \langle f, \theta_n \rangle \, e^{a(x)\sqrt{n^2-\lambda}} \sin nx, \qquad \theta_n = ((I - K(\lambda))^{-1})^* \, \Psi_n.$$

Since $\langle f, \theta_n \rangle = \langle (I - K(\lambda))^{-1} f, \Psi_n \rangle$, the function $G(\lambda)$ is analytic in $\Omega_1 \backslash \mathcal{S}$ because the series

$$\sum_{n=1}^{\infty} \langle (I - K(\lambda))^{-1} f, \Psi_n \rangle \, e^{a(x)\sqrt{n^2-\lambda}} \sin nx$$

converges uniformly in λ on compact subset of $\Omega_1 \backslash \mathcal{S}$. But $G(\lambda) \equiv 0$ for $\lambda \in [0, \omega_0^2]$ (by Theorem 2) and hence

$$G(\lambda) \equiv 0 \qquad \text{on} \quad \Omega_1 \backslash \mathcal{S}$$

by the uniqueness theorem. Hence

$$f(x) = \sum_{n=1}^{\infty} \langle f, \theta_n \rangle \, e^{a(x)\sqrt{n^2-\lambda}} \sin nx$$

for every $\lambda \in \Omega_1 \backslash \mathcal{S}$. Hence by putting $\lambda = \omega^2$, $\omega \in \mathbb{R}$, $\omega^2 \notin \mathcal{S}$ we find that

$$f(x) = \sum_{n=1}^{\infty} \langle f, \theta_n \rangle \, y_n(x)$$

for every $x \in (0, \pi]$.

A direct verification shows that the series on the right converges uniformly on $[\varepsilon, \pi]$ for every $\varepsilon > 0$. ∎

Remark. If the function a is analytic in a neighborhood of $[0, \pi]$, $a(0) = a(\pi) = 0$, $a'(0) \cdot a'(\pi) \neq 0$ and $a(x) < 0$ for $0 < x < \pi$ then there holds the following

THEOREM 4. *If f is analytic in a domain containing $(0, \pi)$, $f(0)=f(\pi)=0$, and $f \in C[0, \pi]$ then*

$$f(x)=\sum_{n=1}^{\infty} \langle f, \theta_n \rangle\, y_n(x)$$

and the series converges uniformly on $[\varepsilon, \pi-\varepsilon]$ for every $\varepsilon \in (0, \frac{\pi}{2})$.

COROLLARY. *If f is analytic in a neighborhood of $[0, \pi]$, $f(0)=f(\pi)=0$, then there holds the conclusion of Theorem* 4.

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